Patterns of Fuzzy Rule-Based Inference

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ABSTRACT

Processing information in fuzzy rule-based systems generally employs one of two patterns of inference: composition or compatibility modification. Composition originated as a generalization of binary logical deduction to fuzzy logic, while compatibility modification was developed to facilitate the evaluation of rules by separating the evaluation of the input from the generation of the output. The first step in compatibility modification inference is to assess the degree to which the input matches the antecedent of a rule. The result of this assessment is then combined with the consequent of the rule to produce the output. This paper examines the relationships between these two patterns of inference and establishes conditions under which they produce equivalent results. The separation of the evaluation of input from the generation of output permits a flexibility in the methods used to compare the input with the antecedent of a rule with multiple clauses. In this case, the degree to which the input and the rule antecedent match is determined by the application of a compatibility measure and an aggregation operator. The order in which these operations are applied may affect the assessment of the degree of matching, which in turn may cause the production of different results. Separability properties are introduced to define conditions under which compatibility modification inference is independent of the input evaluation strategy.

KEYWORDS: ***fuzzy inference, compatibility measures, approximate analogical reasoning, fuzzy if-then rules***

1. INTRODUCTION

Fuzzy set theory provides a formal system suitable for the representation of the vague, imprecise, and ambiguous information that pervades many

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common problem domains. Approximate reasoning using fuzzy techniques has successfully been employed in decision theory, database retrieval, expert systems, and automatic control. The fundamental representation used in many of these applications is that of a fuzzy rule. This paper examines the foundations of the two standard patterns of fuzzy rule-based inference: composition and compatibility modification. Composition originated as a generalization of binary logical deduction to fuzzy logic. Compatibility modification (CM), which was specifically developed for inference in fuzzy rule-based systems, separates the evaluation of the antecedent of a rule from the generation of the output. Compositional inference integrates the analysis of the input and the creation of the output into a single inference step. In CM inference, a compatibility measure is used to determine the degree to which the input matches the antecedent of a rule. After the analysis of the input, the output is constructed from the consequent of the rule and the degree of satisfaction of the antecedent. This paper examines the functions of these two distinct patterns of inference, analyzes their efficiency, and establishes relationships between them.

The examination of fuzzy inference begins with a brief review of the semantics and notation of fuzzy set theory. This is followed by a presentation of the principles of compositional and compatibility modification inference. These inference techniques are then compared, and conditions are established under which they produce the same results. An analysis of the generation of support in CM inference produces the notion of aggregation separability. A compatibility measure and an aggregation operator are said to *aggregation separable* if the same result is obtained regardless of the manner in which the compatibility is evaluated. It is shown that several common families of CM inference techniques satisfy the separability condition.

2. BACKGROUND

A fuzzy set A over domain U is a function $\mu_A : U \to [0,1]$. The function μ_A is called the *membership function* of the fuzzy set. As a generalization of binary logic, a fuzzy set A may be considered to be a predicate. Under this interpretation, $\mu_A(u)$ represents the degree to which A is satisfied by u. An alternative semantics views a fuzzy set as defining a concept. In this case, the membership function expresses the degree to which u matches the criteria that define the concept. Thus, $\mu_A(u)$ indicates the similarity or the compatibility of u with the concept defined by A. The value $\mu_A(u) = 1$ indicates that u is completely compatible with the concept defined by A, while $\mu_A(u) = 0$ indicates u is incompatible with A.

Following the interpretation of fuzzy set theory as an extension of binary logic, the basic fuzzy set operations correspond to the propositional connectives. T-norms provide the fuzzy generalization of conjunction. Formally, a T-norm is a nondecreasing, commutative, and associative function $T:[0,1] \times [0,1] \to [0,1]$ that satisfies $T(0,x) = 0$ and $T(1,x) = x$. Several common T-norms that will be used in the sequel are given in Table 1. Every T-norm T assumes values bounded by T_0 and T_3. That is,

$$T_0(x,y) \le T(x,y) \le T_3(x,y)$$

for all $x, y \in [0,1]$. While many of the results in this paper hold for arbitrary T-norms, we will primarily be concerned with the Łukasiewicz T-norm T_1, the product, and the minimum. Associated with each T-norm is a dual T-conorm S defined by

$$S(x,y) = 1 - T(1-x, 1-y).$$

The T-conorm S is the disjunction that corresponds to the conjunction T. An exposition of the general properties of T-norms can be found in [1–3].

A fuzzy rule specifies an approximate conditional relationship between the elements in a universe U and those in a universe W. The relationship is indicated by a statement of the form "if X is A then Z is C" where the antecedent A is a fuzzy set over U and the consequent C is a fuzzy set over W. Fuzzy rule-based inference combines the relationship with input indicating the current state of U to produce an estimate C' of the state of W. The input is given by a fuzzy set A' over U. The use of fuzzy sets to describe the input permits the representation of imprecision in the specification of the current state of knowledge of U.

A fuzzy set is said to be *normal* if there is at least one element that has the maximal membership value 1. Normality implies that some element in the universe is completely compatible with the concept defined by the fuzzy set. A fuzzy set A is said to be precise if $\mu_A(u_i) = 1$ for some u_i and $\mu_A(u_j) = 0$ for all $j \neq i$. Throughout this paper, A, B, and C will represent fuzzy sets over the domains $U = \{u_1, \dots, u_n\}$, $V = \{v_1, \dots, v_m\}$, and $W = \{w_1, \dots, w_p\}$, respectively.

Table 1. T-Norms and T-Conorms

T-norm	T-conorm
$T_3(x,y) = \min(x,y)$	$S_3(x,y) = \max(x,y)$
$T_2(x,y) = xy$	$S_2(x,y) = x + y - xy$
$T_1(x,y) = \max(x + y - 1, 0)$	$S_1(x,y) = \min(1, x+y)$
$T_0(x,y) = \begin{cases} 1 & \text{if } xy = 1 \\ 0 & \text{otherwise.} \end{cases}$	$S_0(x,y) = \begin{cases} 1 & \text{if } x+y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

3. COMPOSITIONAL INFERENCE

The generalization of inference from bivalued logic to fuzzy logic employs a generalization of the deductive law modus ponens. Following the standard pattern of logical deduction, generalized modus ponens has the form

$$\begin{array}{l} \text{if } X \text{ is } A \text{ then } Z \text{ is } C \\ X \text{ is } A' \\ \hline Z \text{ is } C' \end{array}$$

producing an output fuzzy set C'.

Formally, the relationship specified by the rule "if X is A then Z is C" is given by an implication relation $R_{A \to C}$ over the Cartesian product $U \times W$. An element $R_{A \to C}(u, w)$ indicates the degree to which the truth or presence of u implies that of w. As an extension of logical deduction, $\mu_{C'}(w) = 1$ whenever $\mu_{A'}(u) = 1$ and $R_{A \to C}(u, w) = 1$. A decrease in the certainty of u or of the degree of the implication of w by u is accompanied by a decrease in the implied support for w.

An entry $R_{A \to C}(u, w)$ in the implication relation may be thought of as defining a rule "if u then w" whose strength is $R_{A \to C}(u, w)$. The support for w given input $\mu_{A'}(u)$ is obtained by combining the degree to which u is true with the strength of the implication of w from u. The implied support $T(\mu'_A(u), R_{A \to C}(u, w))$ is the conjunction of $\mu_{A'}(u)$ and $R_{A \to C}(u, w)$, where the T-norm T is called the evaluator of the implication. The most frequently used compositional T-norms are the minimum and the product [4]. These choices provide alternative philosophies for determining support from uncertain inferential information. The standard fuzzy-set conjunction min assigns support based on the weaker of the two conjuncts. Employing the product is similar to support generation with certainty factors [5]: the support for the consequent is affected both by the degree of support for the premise and by the strength of the implication.

The interpretation of the entries $R_{A \to C}(u_i, w)$, $i = 1, \ldots, n$, as rules may be considered to produce a element wise "rule base"

$$\begin{array}{c} \text{If } u_1 \text{ then } w \\ \vdots \\ \text{if } u_n \text{ then } w \end{array}$$

pertaining to w. When input A' is obtained, each rule "if u_i then w" combines with $\mu_{A'}(\mu_i)$ to provide support for w. Following standard conventions, the rules are considered to provide independent support and the consequent w is asserted if it follows from one or more rules. Consequently, the totality of the support for w is obtained by the disjunction of the support by the individual rules. Since the rules are independent and combined disjunctively, the maximum is used to combine the support provided by each of the rules.

A membership value for each $w \in W$ is obtained in the manner described above, producing the resultant fuzzy set C'. This process of determining the support is known as the *compositional rule of inference* (CRI). Formally, the CRI may be written

$$\mu_{C'}(w) = \sup_{u \in U} T(\mu_{A'}(u), R_{A \to C}(u, w)). \tag{1}$$

The compositional rule of inference will also be referred to as *sup-T composition* to explicitly indicate the T-norm that combines the input with the implication relation.

When the fuzzy rule "if X is A then Z is C" is defined by fuzzy sets A and C, the implication relation $R_{A \to C}$ is obtained using an implication operator I, $R_{A \to C}(u, w) = I(\mu_A(u), \mu_C(w))$. Thus compositional inference requires two operations: an implication operator to build the relation, and a T-norm to compose the input with the relation. Many studies [6–15] have examined the properties of the various implication operators, proposed criteria for their selection, and/or performed empirical testing to assess their effectiveness.

4. COMPATIBILITY-MODIFICATION INFERENCE

Compatibility-modification (CM) inference was specifically designed for analyzing information in fuzzy rule-based systems. CM first determines the degree to which the antecedent of a rule is satisfied. This measure of satisfaction is then used to modify the rule's consequent. Dubois and Prade [16, 17] have distinguished CM from CRI inference by calling CM "plausible reasoning." Plausible reasoning may be considered to follow the pattern

$$\begin{array}{l} \text{if } X \text{ is } A \text{ then } Y \text{ is } C \\ X \text{ is } A' \\ A' \text{ is } \gamma \text{ compatible with (similar to) } A \\ \hline C' \text{ is } \gamma \text{ compatible with } C \end{array}$$

in producing the output fuzzy set C'. There are two distinct operations employed in CM inference: measuring the compatibility of the input with the antecedent, and modifying the consequent. When the input is determined to be γ-compatible with the antecedent, the modification produces the fuzzy set C' from the degree of compatibility γ and the consequent C. The term CM inference, which has also been called approximate analogical reasoning [18], is used to encompass all inference techniques that separate the measurement of the compatibility of the antecedent from the modification of the consequent fuzzy set.

The decoupling of the determination of the satisfaction of the antecedent by the input from the generation of the output in CM inference makes it possible to limit the generation of output fuzzy sets to rules whose antecedents match the input to a predetermined degree. Following a strategy often employed in rule-based expert systems, a threshold τ, $0 \leq \tau \leq 1$, may be assigned to each rule. This value represents the degree of agreement required between the antecedent of the rule and the input in order to process the consequent (fire the rule). The possibility of not firing a rule adds significance to the manner in which the compatibility of the input is measured. The consequences of introducing a threshold will be examined in Section 6.

Zadeh's fuzzy interpolation [19] provides an example of CM inference. The compatibility of two fuzzy sets is obtained by a sup-min comparison of the membership values of the elements of the sets. Thus, the compatibility of the input A' with the antecedent A is

$$\gamma = \sup_{u \in U} \min(\mu_{A'}(u), \mu_A(u)). \tag{2}$$

The resulting compatibility measure γ is applied to the consequent fuzzy set C to produce the output fuzzy set C' defined by the membership function

$$\mu_{C'}(w) = \min(\gamma, \mu_C(w)). \tag{3}$$

Thus Zadeh's interpolation is a CM inference technique that uses sup-min to measure the compatibility and the T-norm min to modify the output. Note that, when C is normal, a sup-min comparison of C and C' yields compatibility γ.

Dissimilarity and compatibility measures are often considered interchangable, since a compatibility measure may be constructed from dissimilarity and vice versa. Given a dissimilarity measure *dis* whose maximal value is 1, then $1 - dis(A, A')$ is a measure of the compatibility of A and A'. Employing the relationship between compatibility and dissimilarity, Yager [20] formalized CM inference based on a dissimilarity measure dis and an implication operator I. The dissimilarity of A and A' is used to modify C by $\mu_{C'}(w) = I(\text{dis}(A, A'), \mu_C(w))$. Soula [21] proposed the Hamming distance between A and A', and Dubois and Prade [16] suggested the Hausdorff metric for measuring the dissimilarity of fuzzy sets for CM inference.

Magrez and Smets [13] introduced a CM technique that measures the compatibility of A and A' as the necessity of A knowing that A' is true [22],

$$\delta = \sup_{u \in U} \min(1 - \mu_A(u), \mu_{A'}(u)) = \inf_{u \in U} \max(\mu_A(u), 1 - \mu_{A'}). \quad (4)$$

The compatibility measure is subtracted from 1 to produce a dissimilarity measure $1 - \delta$, which is then added (using bounded-sum addition) to the membership value of each element in the consequent C,

$$\mu_{C'}(w) = \min(1, \mu_C(w) + 1 - \delta). \quad (5)$$

Thus, this technique falls within the class of CM inference described by Yager. The computation given in Equation (5) may also be obtained by applying the Łukasiewicz implication to $\mu_C(w)$ and δ.

5. CRI AND CM

Compositional inference requires the construction of an implication relation $R_{A \to C}$ over $U \times W$ whose values quantify the degree to which an element in $w \in W$ is implied by the presence of an element $u \in U$. Two distinct approaches are commonly employed to produce the implication relation $R_{A \to C}$ from the antecedent and consequent fuzzy sets in the rule "if X is A then Z is C." One approach constructs the implication relation using a T-norm to combine the antecedent and the consequent while the other employs an implication operator. This section develops the relationships between CRI and CM inference using these methods of constructing the implication relation.

When a T-norm is used to construct the implication $R_{A \to C}$, the entries are given by $R_{A \to C}(u, w) = T(u, w)$. This type of sup-$T$ compositional inference was proposed in early fuzzy control systems [23] and continues to be the technique of choice for automatic-control applications.

For precise input, fuzzy interpolation [Equations (2) and (3)] has been shown to be equivalent to sup-min composition [4, 24]. Turksen [24] extended this equivalence to the case when an arbitrary T-norm is used for both the CRI implication operator and the interpolation modification function. Proposition 1 further extends the equivalence of sup-T composition and CM inference to the case of fuzzy input. The generalization of interpolation that employs sup-T to measure the compatibility and modifies the consequent with same T-norm T will be called sup-T CM.

PROPOSITION 1.0 *For every fuzzy input A',* sup-T *CM is equivalent to* sup-T *composition when $R_{R \to C}(u, w) = T(u, w)$.*

Proof The result is shown for the rule "if X is A then Z is C" and input A'. The membership value for an element w_k obtained by processing the rule with the sup-T CM is

$$\mu_{C'}(w_k) = T\Big(\sup_i T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)\Big). \tag{6}$$

Let u_l be an element for which $\sup_i T(\mu_{A'}(u_i), \mu_A(u_i)) = T(\mu_{A'}(u_l), \mu_A(u_l))$. That is, u_l is an element of U that exhibits the maximal T compatibility of A and A'. By the monotonicity of T-norms,

$$T(T(\mu_{A'}(u_l), \mu_A(u_l)), \mu_C(w_k)) \geq T(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k))$$

for all i, so that

$$T(T(\mu_{A'}(u_l), \mu_A(u_l)), \mu_C(w_k)) = \sup_i T(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)). \tag{7}$$

Substituting $\sup_i T(\mu_{A'}(u_i), \mu_A(u_i))$ for $T(\mu_A(u_l), \mu_{A'}(u_l))$ in (7) yields

$$T\Big(\sup_i T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)\Big) = \sup_i T(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)) \tag{8}$$

Combining (8) and (6), we get

$$\mu_{C'}(w) = \sup_i T(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)).$$

The fuzzy set C'' obtained by the sup-T composition of A' with $R_{A \to C}(u, w) = T(u, w)$ has membership values

$$\begin{aligned}\mu_{C''}(w_k) &= \sup_i T(\mu_{A'}(u_i), T(\mu_A(u_i), \mu_C(w_k))) \\ &= \sup_i T(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_G(w_k)),\end{aligned}$$

where the final step follows by the associativity of T-norms. Thus $\mu_{C'} = \mu_{C''}$ and the proof is complete. ■

An alternative approach to the construction of an implication relation is to directly translate propositional implication to fuzzy predicates. That is, the implication operator is obtained from a fuzzy disjunction and negation operator according to the propositional tautology $A \to C \equiv \neg A$ or C. Under this interpretation,

$$I(x, y) = S(1 - x, y) = 1 - T(x, 1 - y), \tag{9}$$

where S is a T-conorm and T is the corresponding dual T-norm. An implication operator constructed from (9) is called an S-implication. The original and most widely used form of composition [25, 19] uses Łukasiewicz implication $I(x, y) = \max(1, 1 - x + y)$ with min combining the input with the implication.

As before, consider the rule "if X is A then Z is C" with input A'. The output membership value for an element w_k obtained by processing the rule with sup-T CM using an implication operator I for the modification function is

$$\begin{aligned}
\mu_{C'}(w_k) &= I\Big(\sup_i T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)\Big) \\
&= \inf_i I(T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)) \\
&= \inf_i S(1 - T(\mu_{A'}(u_i), \mu_A(u_i)), \mu_C(w_k)) \\
&= \inf_i S(S(1 - \mu_{A'}(u_i), 1 - \mu_A(u_i)), \mu_C(w_k)) \\
&= \inf_i S(1 - \mu_{A'}(u_i), S(1 - \mu_A(u_i), \mu_C(w_k))). \qquad (10)
\end{aligned}$$

The result obtained by sup-T compositional inference with the relation $R_{A \to C}$ constructed using an S-implication operator I is

$$\begin{aligned}
\mu_{C''}(w_k) &= \sup_i T(\mu_{A'}(u_i), I(\mu_A(u_i), \mu_C(w_k))) \\
&= \sup_i T(\mu_{A'}(u_i), S(1 - \mu_A(u_i), \mu_C(w_k))). \qquad (11)
\end{aligned}$$

Unlike the situation when a T-norm is used to construct the implication relation, the CM and CRI inference are not equivalent when the relation is obtained from an S-implication operator. Although these two methods do not produce the same values, Proposition 2 shows that the fuzzy set produced by CM is a subset of that produced by CRI.

PROPOSITION 2.0 *Let T be a T-norm, S be the associated T-conorm, and I be the S-implication operator $I(x, y) = S(1 - x, y)$. Also let C' be defined by* (10) *and C'' be defined by* (11). *If input fuzzy set A' is normal, then $C' \subseteq C''$.*

Proof Let u_j be an element of U for which $\mu_{A'}(u_j) = 1$. From (10),

$$\begin{aligned}
\mu_{C'}(w_k) &= \inf_i S(1 - \mu_{A'}(u_i), S(1 - \mu_A(u_i), \mu_C(w_k))) \\
&\le S\big(1 - \mu_{A'}(u_j), S\big(1 - \mu_A(u_j), \mu_C(w_k)\big)\big) \\
&= S\big(0, S\big(1 - \mu_A(u_j), \mu_C(w_k)\big)\big) \\
&= S\big(1 - \mu_A(u_j), \mu_C(w_k)\big).
\end{aligned}$$

Now, from (11),

$$\begin{aligned}\mu_{C''}(w_k) &= \sup_i T(\mu_{A'}(u_i), S(1 - \mu_A(u_i), \mu_C(w_k)))\\ &\geq T\big(\mu_{A'}(u_j), S\big(1 - \mu_A(u_j), \mu_C(w_k)\big)\big)\\ &= T\big(1, S\big(1 - \mu_A(u_j), \mu_C(w_k)\big)\big)\\ &= S\big(1 - \mu_A(u_j), \mu_C(w_k)\big).\end{aligned}$$

Thus $\mu_{C'}(w_k) \leq \mu_{C''}(w_k)$ for every $w_k \in W$ and $C' \subseteq C''$. ■

The fuzzy sets A and A' over $U = \{u_1, u_2, u_3\}$ defined by the membership functions

	u_1	u_2	u_3
A	0	1	0.5
A'	1	0.5	0.25

demonstrate that it is not necessarily the case that $C' = C''$. In this example, max and min are used as the T-norm and T-conorm. Consider an element in $w \in W$ with $\mu_C(w) = 0.5$. Then $\mu_{C'}(w) = 0.5$ and $\mu_{C''}(w) = 1$. The normality of A' is essential for the relationship demonstrated in Proposition 2. It is easy to construct fuzzy sets A, A', and C for which $\mu_{C'}(w) > \mu_{C''}(w)$ when A' is subnormal.

6. PROPERTIES OF CM INFERENCE

One important advantage of the separation of the analysis of input from the generation of output is the efficiency gained by eliminating the need to produce output for every rule. The separation permits the introduction of a threshold τ to specify the minimal degree of satisfaction of the antecedent of the rule necessary for the rule to fire. The possibility of not firing a rule adds significance to the manner in which the compatibility of the input is measured.

Further efficiency may be gained by a modular evaluation of compatibility when the antecedent contains multiple-clauses. There are two distinct strategies that may be employed to determine the degree to which the input matches the antecedent. These strategies have been named *aggregation-compatibility* and *compatibility-aggregation* evaluation: the name describes the order of application of the operations used to determine the degree to which the input data match the antecedent of a rule. There is no distinction between these techniques when the antecedent consists of a single clause. When the degree of matching is independent of the order in which the operations are performed, the compatibility and aggregation operators are said to be *separable* [26]. The importance of separability in the processing of input is discussed in Section 6.1. This is followed by establishing separability conditions for several common combinations of compatibility measures and aggregators.

6.1. Compatibility and Aggregation

A rule "if X is A and Y is B then Z is C" represents a relationship between the values of the input domains U and V and the output domain W. The CM evaluation of such a rule inference requires a measure of the degree to which the input matches the antecedent. One strategy is to transform the complex rule into a rule whose antecedent consists of a single fuzzy set. Once this has been accomplished, the CM strategies introduced in Section 4 may be applied to this new rule.

To obtain a rule in which the antecedent consists of a single clause, the fuzzy sets A and B are combined to produce a fuzzy relation over $U \times V$. The relation $R_{A \times B}$ is constructed by combining the information in each of the individual sets by a T-norm $R_{A \times B}(u_i, v_j) = T(\mu_A(u_i), \mu_B(v_j))$. Using the relation $R_{A \times B}$, the original rule may be considered to have the form "if (X, Y) is $R_{A \times B}$ then Z is C," whose antecedent is the Cartesian product $U \times V$. (Generally, this approach assumes the antecedent clauses to be noninteractive and uses the T-norm min.) The input, fuzzy sets A' and B' describing the states of U and V, is combined in the same manner, producing a relation $R_{A' \times B'}(u_i, v_j)$.

After the construction of the relations, a compatibility measure *com* is used to determine the compatibility $com(R_{A \times B}, R_{A' \times B'})$ of the input with the antecedent relation. As before, the compatibility is then used to modify the consequent. This process is called *aggregation-compatibility* CM, since it first combines (aggregates) the clauses in the antecedent to produce a relation and then compares the input with the resulting relation. Letting $\cap$ represent the aggregator and com the compatibility measure, aggregation-compatibility determines the compatibility by evaluating com$(A \cap B, A' \cap B')$. Figure 1 illustrates the sequence in which the operations are performed to produce the input and obtain the compatibility in aggregation-compatibility CM.

In CM inference, there is no specific requirement for the construction of the relation $R_{A \times B}$. When input A' and B' are acquired, the compatibility between A and A', $com(A, A')$, and the compatibility between B and B', $com(B, B')$, may be directly evaluated (see Figure 2). The two compatibilities are combined using an aggregation operator $\oplus$ to obtain the overall compatibility between the antecedent and the input. This modular ap-

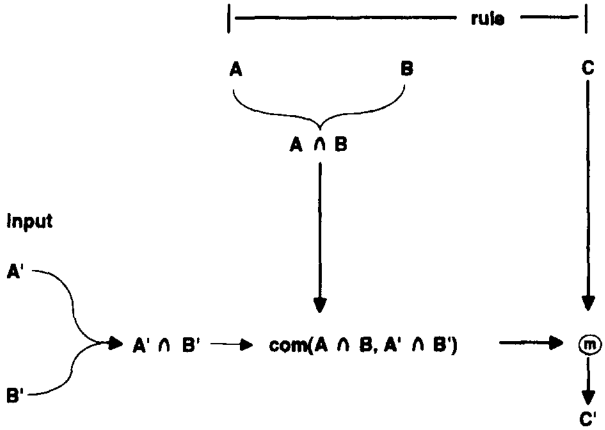

Figure 1. Aggregation-compatibility evaluation.

proach to compatibility evaluation is called *compatibility-aggregation* CM. In terms of the operations involved, a compatibility-aggregation assessment of the input and the antecedent is obtained by evaluating $\oplus(\mathrm{com}(A, A'), \mathrm{com}(B, B'))$.

The order of the operations has a significant effect on the efficiency of evaluating compatibility. Using the compatibility-aggregation approach, measuring compatibility requires $n + m$ operations (recall that n and m are the cardinalities of U and V, respectively). A single application of the aggregation operator combines the two compatibility measures. In the aggregation-compatibility method, producing the input fuzzy relation requires nm operations, and the measure of the compatibility is also a function of the size of the relation.

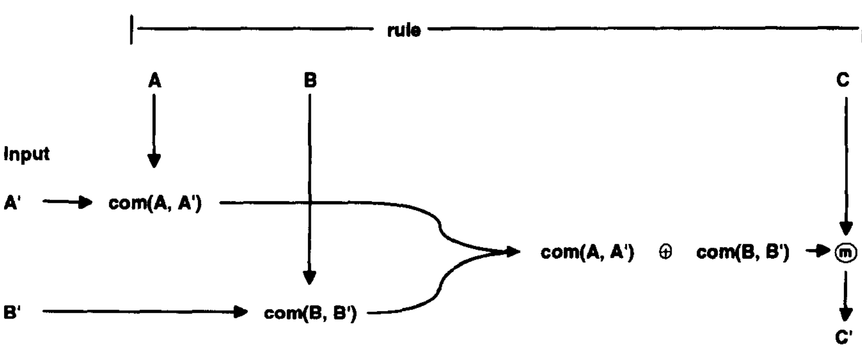

Figure 2. Compatibility-aggregation evaluation.

6.2. Aggregation Separability

The two distinct methods of evaluating the satisfaction of the antecedent of a rule illustrated in Figures 1 and 2 may produce different values. This section introduces the notion of aggregation separability and investigates the relationships between aggregation-compatibility and compatibility-aggregation evaluation.

DEFINITION 0.1 *A compatibility measure com and aggregator* $\oplus$ *are said to be*

(i) greater than aggregation separable if

$$com(A \cap B, A' \cap B') \geq \oplus(com(A, A'), com(B, B')), \quad (12)$$

(ii) less than aggregation separable if

$$com(A \cap B, A' \cap B') \leq \oplus(com(A, A'), com(B, B')), \quad (13)$$

(iii) separable if they are both less than and greater than aggregation separable.

Bilgic and Turksen [27] suggested that greater than separability is a desirable property for fuzzy rule-based systems, since it guarantees that whenever the compatibility-aggregation evaluation of the input exceeds a threshold, the aggregation-compatibility evaluation does also. However, there may be input A' and B' for which com($A \cap B$, $A' \cap B'$) $\geq \tau >$ $\oplus$(com(A, A'), com(B, B')), creating a situation where the aggregation-compatibility evaluation would cause the rule's consequent to be processed but the compatibility-aggregation method would not. When greater than separability is satisfied, a two-step procedure for firing rules has been proposed: if $\oplus$(com(A, A'), com(B, B')) $\geq \tau$, the consequent is automatically processed; otherwise, com($A \cap B$, $A' \cap B'$) is computed to decide whether to process the consequent.

Less than separability is briefly mentioned in [27], but no complete description of its usefulness is provided. This property might be considered desirable in that it would guarantee that whenever the compatibility-aggregation evaluation did not fire a rule, the aggregation-compatibility evaluation would not either. The satisfaction of (13), however, does not eliminate situations where the threshold τ falls between com($A \cap B$, $A' \cap B'$) and $\oplus$(com(A, A'), com(B, B')). In this case, the compatibility-aggregation evaluation would fire the rule, but aggregation-compatibility would not. With less than separability, an algorithm for processing rules could test if $\oplus$(com(A, A'), com(B, B')) $< \tau$, in which case the consequent would not be processed; otherwise, com($A \cap B$, $A' \cap B'$) must be computed to decide if the consequent is processed.

The premise underlying the two rule-firing strategies discussed above is that aggregation-compatibility is the accepted method of evaluation of rules with complex antecedents and that compatibility-aggregation is an efficient method for approximating the compatibility. The satisfaction of separability conditions provides a relationship between these two forms of inference. The following sections present combinations of compatibility measures and aggregators that satisfy the separability conditions.

6.3. Separability and Partial Matching

First we consider the separability of one of the most popular family of compatibility measures, the consistency or partial-matching indices. A partial matching index is defined by

$$\mathrm{PM}(A, A') = \sup_i T(\mu_A(u_i), \mu_{A'}(u_i)) \tag{14}$$

where T is any T-norm. Partial matching indices provide an optimistic evaluation of compatibility: the degree to which two fuzzy sets match is determined by the single element which individually has the most agreement. Properties of partial-matching indices are presented and analyzed in [22, 28]. Proposition 3 establishes conditions under which PM compatibility is separable. Throughout this section, we let $a_i = \mu_A(u_i)$ and $a'_i = \mu_{A'}(u_i)$.

PROPOSITION 3.0 *Let* PM *be a partial matching index defined by the T-norm T. Then*

$$\mathrm{PM}(A \cap B, A' \cap B') = T(\mathrm{PM}(A, A'), \mathrm{PM}(B, B')) \tag{15}$$

when the aggregation operator $\cap$ is T.

Proof Rewriting the left-hand side of (15) by substituting the definition of PM yields

$$\begin{aligned} PM(A \cap B, A' \cap B) &= \sup_{ij} T\big(T(a_i, b_j), T(a'_i, b'_j)\big) \\ &= \sup_{ij} T\big(T(a_i, a'_i), T(b_j, b'_j)\big). \end{aligned}$$

The final step follows from the associativity of T-norms.

Now let k be the subscript for which $\sup_i T(a_i, a'_i) = T(a_k, a'_k)$, that is, $T(a_k, a'_k) \geq T(a_i, a'_i)$ for $1 \leq i \leq n$. Similarly, let $\sup_j T(b_j, b'_j) = T(b_l, b'_l)$. By the monotonicity of T-norms,

$$T(T(a_k, a'_k), T(b_l, b'_l)) \geq T\big(T(a_i, a'_i), T(b_j, b'_j)\big)$$

for $1 \le i \le n$ and $1 \le j \le m$. The right-hand side of (15) may be written

$$\begin{aligned} T(PM(A, A'), PM(B, B')) &= T\left(\sup_i T(a_i, a_i'), \sup_j T(b_j, b_j')\right) \\ &= T(T(a_k, a_k'), T(b_l, b_l')) \\ &= \sup_{ij} T\big(T(a_i, a_i'), T(b_j, b_j')\big). \end{aligned}$$

Thus, (15) holds when a single T-norm T is used for both the partial matching index and the aggregator. ∎

It is reasonable to assume that the same T-norm should be used for both the T and the $\cap$ in (15), since this T-norm represents a logical aggregation of the antecedent clauses, the only difference being when the aggregation is performed. On the left-hand side of (15), the aggregation is performed before the compatibility is evaluated. On the right-hand side, the aggregation is done after the independent evaluation of the compatibility of each clause in the antecedent. It does not follow, however, that the T-norm in the definition of the partial matching index must be the same T-norm that is used for the aggregation. This flexibility permits the definition of the compatibility measure to be independent of that of the aggregator.

Pursuing this latter possibility, we show that separability is not guaranteed when the T-norm in the PM compatibility measure differs from the one used for both the $\cap$ and T in (15). Choosing T_2 to be the aggregation operator and sup-T_3 as the compatibility measure produces

$$\mathrm{PM}(A \cap B, A' \cap B') = \sup_{ij} T_3\big[T_2(a_i', b_j'), T_2(a_i, b_j)\big] \tag{16}$$

and

$$T(\mathrm{PM}(A, A'), \mathrm{PM}(B, B')) = T_2\left[\sup_i T_3(a_i', a_i), \sup_j T_3(b_j', b_j)\right]. \tag{17}$$

A simple counterexample shows that (16) and (17) are not equivalent. Table 2 gives the values of fuzzy sets, A, B, A', and B' and the pairwise T_2 and T_3 combinations of A and A' and B and B'. The value of (17) is

Table 2. Example for Interchanging T-Norms

i	A	B	A'	B'	$T_3(a_i, a_i')$	$T_3(b_j, b_j')$	$T_2(a_i, a_i')$	$T_2(b_j, b_j')$
1	0.7	0.0	0.0	0.6	0.0	0.0	0.0	0.0
2	1.0	0.5	0.3	1.0	0.3	0.5	0.3	0.5
3	0.8	1.0	1.0	0.7	0.8	0.7	0.8	0.7
4	0.0	0.5	0.4	0.0	0.0	0.0	0.0	0.0
sup					0.8	0.7	0.8	0.7

$T_2(0.8, 0.7) = 0.56$. For this assignment input A' and B', $\sup_{ij} T_3[T_2(a_i', b_j'), T_2(a_i, b_j)] = 0.7$. Thus this combination of compatibility measure and aggregator is not less than aggregation separable.

Interchanging T_3 and T_2 in (16) and (17) produces

$$\mathrm{PM}(A \cap B, A' \cap B') = \sup_{ij} T_2\left[T_3(a_i', b_j'), T_3(a_i, b_j)\right] = 0.56 \quad (18)$$

and

$$T(\mathrm{PM}(A, A'), \mathrm{PM}(B, B'))$$
$$= T_3\left[\sup_i T_2(a_i', a_i), \sup_j T_2(b_j', b_j)\right] = 0.7, \quad (19)$$

respectively. This gives an example of a selection of T-norms that do not satisfy greater than aggregation separability. The counterexamples show that when different T-norms are used for the partial-matching index and for the aggregator, neither type of separability is ensured.

Though not presented in this manner, aggregation separability is addressed in [4]. Lee is concerned with determining whether sup-T compositional inference with the aggregation of the clauses produces the same result as performing an independent sup-T composition for each clause and then aggregating the results. In Lee's presentation, the analysis was not restricted to measuring the compatibility of the antecedent with the input, but rather examined the output that was produced. However, when the output domain $W = \{w\}$ consists of a single element with $\mu_C(w) = 1.0$, sup-T composition is equivalent to determining the PM compatibility (14). Lemmas 2 and 2′ in [4] suggest that (15) should hold when the T-norm used to determine the compatibility differs from that used to aggregate the information. A counterexample to this assertion has been provided above. The results given in [4] hold only when the same T-norm is used throughout the expression.

6.4. Less Than Separability and Necessity

One way to determine the applicability of a rule is to measure the degree to which the antecedent is necessary given the input. The necessity of a fuzzy set A given A' is defined by

$$\mathrm{nec}(A', A) = \inf_i S(1 - \mu_{A'}(u_i), \mu_A(u_i)),$$

where S is a T-conorm. With necessity measuring the compatibility, less than aggregation separability holds when the aggregator is a T-conorm.

PROPOSITION 4.0 *Let S be a T-conorm, T the associated T-norm, and nec the necessity measure constructed from S. Under these conditions, less than aggregation separability holds when* $\cap = T$ *in* $A \cap B$ *and* $\oplus = S$.

Proof We must show that

$$\operatorname{nec}(A' \cap B', A \cap B) \leq \oplus(\operatorname{nec}(A', A), \operatorname{nec}(B', B)) \tag{20}$$

with $\cap = T$ and $\oplus = S$. Rewriting the right-hand side of (20) in terms of the T-conorm S produces

$$\begin{aligned}\oplus(\operatorname{nec}(A', A), \operatorname{nec}(B', B)) &= S\Big(\inf_i S(1 - a_i', a_i), \inf_j S(1 - b_j', b_j)\Big) \\ &= \inf_{ij} S\big(S(1 - a_i', a_i), S(1 - b_j', b_j)\big) \\ &= \inf_{ij} S\big(S(1 - a_i', 1 - b_j'), S(a_i, b_j)\big). \end{aligned} \tag{21}$$

The second and third steps follow by monotonicity and associativity of T-conorms. But

$$\operatorname{nec}(A' \cap B', A \cap B) = \inf_{ij} S\big(S(1 - a_i', 1 - b_i'), T(a_i, b_j)\big),$$

which is less than (21) by the ordering between T-norms and T-conorms and the monotonicity of T-conorms. ∎

6.5. Greater Than Separability and Dissimilarity

The class of CM inference techniques defined by Yager (Section 2) uses dissimilarity rather than compatibility to measure the degree of matching between the antecedent and the input. The dissimilarity formulation of greater than aggregation separability is

$$\oplus(\mathit{diss}(A, A'), \mathit{diss}(B, B')) \geq \mathit{diss}(A \cap B, A' \cap B'), \tag{22}$$

where diss is a dissimilarity measure and $\oplus$ is an aggregator. In fuzzy inference using dissimilarity measures, a threshold τ indicates the lower bound on the dissimilarity of the rule's antecedent with the input. That is, a rule will be fired only when the dissimilarity is less than τ. The satisfaction of (22) guarantees than whenever the dissimilarity-aggregation evaluation of the input does not exceed the threshold, aggregation-dissimilarity will not either.

Dissimilarity measures for fuzzy sets can be obtained by considering a fuzzy set over a universe $U = \{u_1, \ldots, u_n\}$ as a point in $\mathbf{R}^n$. A fuzzy set A over U is represented by the point $[\mu_A(u_1), \ldots, \mu_A(u_n)]$. The distance between two fuzzy sets (points in $\mathbf{R}^n$) can be considered to be an estimate of their similarity. Thus a metric on $\mathbf{R}^n$ provides a dissimilarity measure: identical fuzzy sets are assigned measure 0, and the measure increases with the difference between the sets. With this interpretation, the normalized Minkowski r-metrics, defined by

$$\frac{d_r(A, A')}{n^{1/r}} = \frac{\left(\sum_{i=1}^{n} |\mu_A(u_i) - \mu_{A'}(u_i)|^r\right)^{1/r}}{n^{1/r}} \tag{23}$$

with $r \geq 1$, provide a family of fuzzy-set dissimilarity measures. It has been shown [28] that rule-based inference using a dissimilarity measure derived from the normalized d_r-metrics with addition for $\oplus$ and T_1 or T_3 for $\cap$ satisfies (22).

The compatibility measure associated with a normalized Minkowski dissimilarity measure is $1 - d_r/n^{1/r}$. Substituting this compatibility measure and addition for $\oplus$ into the definition of greater than aggregation separability (12) produces

$$1 - \frac{d_r(A \cap B, A' \cap B')}{(nm)^{1/r}} \tag{24}$$

for the left-hand side of (12), and

$$1 - \frac{d_r(A, A')}{n^{1/r}} + 1 - \frac{d_r(B, B')}{m^{1/r}} \tag{25}$$

for the right-hand side. The fuzzy sets given in Table 3 show that the compatibility measure obtained from d_1 does not satisfy greater than aggregation separability. The expression for the left-hand side evaluates to 0.5, and the right-hand side to 1.5.

The preceding shows that the standard conversion between compatibility and dissimilarity does not preserve separability. When using a threshold to determine the applicability of rules, the choice of using a dissimilarity measure or a compatibility measure may alter the set of rules to be fired.

Table 3. Example for Dissimilarity

Element	A	B	A'	B'
1	1.0	1.0	1.0	0.5
2	1.0	1.0	1.0	0.5

7. CONCLUSION

Composition and compatibility modification provide two patterns of inference for fuzzy rule-based systems. Compatibility modification permits the introduction of a threshold for determining the applicability of rules, thereby reducing the production of output. Additional efficiency can be achieved by the modular evaluation of the compatibility of the input with the antecedent of the rule. It has been shown that the manner in which the compatibility is measured may alter the set of applicable rules. The satisfaction of the separability conditions provides a strategy for the selection of the rules to be fired that takes advantage of the efficiency gains incurred by modular compatibility evaluation.

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